

The single equality $A^{*n}A^n = (A^*A)^n$ does not imply the quasinormality of weighted shifts on rootless directed trees

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ABSTRACT. It is proved that each bounded injective bilateral weighted shift W satisfying the equality $W^{*n}W^n = (W^*W)^n$ for some integer $n \geq 2$ is quasinormal. For any integer $n \geq 2$, an example of a bounded non-quasinormal weighted shift A on a rootless directed tree with one branching vertex which satisfies the equality $A^{*n}A^n = (A^*A)^n$ is constructed. It is also shown that such an example can be constructed in the class of composition operators in L^2 -spaces over σ -finite measure spaces.

1. Introduction

The class of bounded quasinormal operators was introduced by A. Brown in [2]. Two different definitions of unbounded quasinormal operators appeared independently in [12] and in [16]. As recently shown in [10], these two definitions are equivalent. Following [16], we say that a closed densely defined operator A in a complex Hilbert space \mathcal{H} is *quasinormal* if A commutes with the spectral measure E of $|A|$, i.e. $E(\sigma)A \subset CE(\sigma)$ for all Borel subsets σ of the nonnegative part of the real line. By [16, Proposition 1], a closed densely defined operator A in \mathcal{H} is quasinormal if and only if $U|A| \subset |A|U$, where $A = U|A|$ is the polar decomposition of A (cf. [17, Theorem 7.20]). It is well-known that quasinormal operators are always subnormal and that the reverse implication does not hold in general. Yet another characterization of quasinormality of unbounded operators states that a closed densely defined operator A is quasinormal if and only if the equality $A^{*n}A^n = (A^*A)^n$ holds for $n = 2, 3$ (see [10, Theorem 3.6]; see also [11] for the case of bounded operators and [5, p. 63] for a prototype of this characterization). For more information on quasinormal operators we refer the reader to [2, 4], the bounded case, and to [12, 16, 13, 10], the unbounded one.

In view of the above discussion, the question arises as to whether the single equality $A^{*n}A^n = (A^*A)^n$ with $n \geq 2$ implies the quasinormality of A . It turns out that the answer to this question is in the negative. In fact, as recently shown in [10, Example 5.5], for every integer $n \geq 2$, there exists a weighted shift A on a rooted and leafless directed tree with one branching vertex such that

$$(1.1) \quad (A^*A)^n = A^{*n}A^n \text{ and } (A^*A)^k \neq A^{*k}A^k \text{ for all } k \in \{2, 3, \dots\} \setminus \{n\}.$$

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It remained an open question as to whether such construction is possible on a rootless and leafless directed tree. This is strongly related to the question of the existence of a composition operator A in an L^2 -space (over a σ -finite measure space) which satisfies (1.1). In this paper, we will construct for every integer $n \geq 2$ examples of bounded (necessarily non-quasinormal) weighted shifts A on a rootless and leafless directed tree with one branching vertex which satisfy (1.1) (cf. Theorem 5.3). This combined with the fact that every weighted shift on a rootless directed tree with nonzero weights is unitarily equivalent to a composition operator in an L^2 -space (see [8, Theorem 3.2.1] and [9, Lemma 4.3.1]) yields examples of composition operators satisfying (1.1) (cf. Theorem 5.3).

It was observed in [10, p. 144] that a unilateral or a bilateral injective weighted shift W which satisfies the equality $(W^*W)^k = (W^*)^k W^k$ for $k = 2$ is quasinormal, and that the same is true for $k = 3$ provided W is bounded. In the present paper we will show that in the class of bounded injective bilateral weighted shifts, the single equality $W^{*n}W^n = (W^*W)^n$ with $n \geq 2$ does imply quasinormality (cf. Theorem 4.3). This is no longer true for unbounded ones even for $k = 3$ (cf. Example 4.4).

2. Preliminaries

In this paper we use the following notation. The fields of rational, algebraic, real and complex numbers are denoted by \mathbb{Q} , \mathbb{A} , \mathbb{R} and \mathbb{C} , respectively. The symbols \mathbb{Z} , \mathbb{Z}_+ , \mathbb{N} and \mathbb{R}_+ stand for the sets of integers, nonnegative integers, positive integers and nonnegative real numbers, respectively. The field of rational functions in x with rational coefficients is denoted by $\mathbb{Q}(x)$. We write $\mathbb{Z}[x]$ for the ring of all polynomials in x with integer coefficients.

Let A be a linear operator in a complex Hilbert space \mathcal{H} . Denote by $\mathcal{D}(A)$, \bar{A} and A^* the domain, the closure and the adjoint of A respectively (provided they exist). A subspace \mathcal{E} of $\mathcal{D}(A)$ is called a *core* for A if \mathcal{E} is dense in $\mathcal{D}(A)$ with respect to the graph norm of A . We write $\mathbf{B}(\mathcal{H})$ for the set of all bounded operators in \mathcal{H} whose domain are equal to \mathcal{H} .

In the present paper, by a *classical weighted shift* we mean either a unilateral weighted shift W in ℓ^2 or a bilateral weighted shift W in $\ell^2(\mathbb{Z})$. To be more precise, W is understood as the product VD , where, in the unilateral case, V is the unilateral isometric shift on ℓ^2 of multiplicity 1 and D is a diagonal operator in ℓ^2 with diagonal elements $\{\lambda_n\}_{n=0}^\infty$; in the bilateral case, V is the bilateral unitary shift on $\ell^2(\mathbb{Z})$ of multiplicity 1 and D is a diagonal operator in $\ell^2(\mathbb{Z})$ with diagonal elements $\{\lambda_n\}_{n=-\infty}^\infty$. In fact, W is a unique closed linear operator in ℓ^2 (respectively, $\ell^2(\mathbb{Z})$) such that the linear span of the standard orthonormal basis $\{e_n\}_{n=0}^\infty$ of ℓ^2 (respectively, $\{e_n\}_{n=-\infty}^\infty$ of $\ell^2(\mathbb{Z})$) is a core for W and

$$(2.1) \quad We_n = \lambda_n e_{n+1} \quad \text{for } n \in \mathbb{Z}_+ \quad (\text{respectively, } n \in \mathbb{Z}).$$

Suppose $\mathcal{T} = (V; E)$ is a directed tree (V and E are the sets of vertices and edges of \mathcal{T} , respectively). If \mathcal{T} has a root, we denote it by root . Put $V^\circ = V \setminus \{\text{root}\}$ if \mathcal{T} has a root and $V^\circ = V$ otherwise. For every $u \in V^\circ$, there exists a unique $v \in V$, denoted by $\text{par}(u)$, such that $(v; u) \in E$. For any vertex $u \in V$ we put $\text{Chi}(u) = \{v \in V : (u, v) \in E\}$. The Hilbert space of square summable complex functions on V equipped with the standard inner product is denoted by $\ell^2(V)$. For $u \in V$, we define $e_u \in \ell^2(V)$ to be the characteristic function of the one-point set $\{u\}$.

Given a system $\lambda = \{\lambda_v\}_{v \in V^\circ}$ of complex numbers, we define the operator S_λ in $\ell^2(V)$, which is called a *weighted shift* on \mathcal{T} with weights λ , as follows

$$\mathcal{D}(S_\lambda) = \{f \in \ell^2(V) : \Lambda_{\mathcal{T}} f \in \ell^2(V)\} \quad \text{and} \quad S_\lambda = \Lambda_{\mathcal{T}} f \quad \text{for} \quad f \in \mathcal{D}(S_\lambda),$$

where

$$(\Lambda_{\mathcal{T}} f)(v) = \begin{cases} \lambda_v f(\text{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{otherwise.} \end{cases}$$

We refer the reader to [8] for more details on weighted shifts on directed trees and their relations to classical weighted shifts.

Let us recall some useful properties of weighted shifts on directed trees we need in this paper.

PROPOSITION 2.1 ([8, Proposition 3.1.3]). *Let S_λ be a weighted shift on a directed tree \mathcal{T} with weights $\lambda = \{\lambda_v\}_{v \in V^\circ}$. Then the following assertions hold:*

- (i) S_λ is a closed operator,
- (ii) $e_u \in \mathcal{D}(S_\lambda)$ if and only if $\sum_{v \in \text{Chi}(u)} |\lambda_v|^2 < \infty$ and in this case

$$S_\lambda e_u = \sum_{v \in \text{Chi}(u)} \lambda_v e_v, \quad \|S_\lambda e_u\| = \sum_{v \in \text{Chi}(u)} |\lambda_v|^2,$$

- (iii) S_λ is densely defined if and only if $e_u \in \mathcal{D}(S_\lambda)$ for every $u \in V$.

PROPOSITION 2.2 ([8, Proposition 3.1.8]). *Let S_λ be a weighted shift on a directed tree \mathcal{T} with weights $\lambda = \{\lambda_u\}_{u \in V^\circ}$. Then the following conditions are equivalent:*

- (i) $\mathcal{D}(S_\lambda) = \ell^2(V)$,
- (ii) $S_\lambda \in \mathbf{B}(\ell^2(V))$,
- (iii) $\sup_{u \in V} \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 < \infty$.

Moreover, if $S_\lambda \in \mathbf{B}(\ell^2(V))$, then

$$\|S_\lambda\| = \sup_{u \in V} \|S_\lambda e_u\| = \sqrt{\sup_{u \in V} \sum_{v \in \text{Chi}(u)} |\lambda_v|^2}.$$

PROPOSITION 2.3 ([8, Proposition 8.1.7]). *Let $n \in \mathbb{Z}_+$. If $S_\lambda \in \mathbf{B}(\ell^2(V))$ is a weighted shift on a directed tree \mathcal{T} with weights $\lambda = \{\lambda_v\}_{v \in V^\circ}$, then the following two conditions are equivalent:*

- (i) $(S_\lambda^* S_\lambda)^n = (S_\lambda^*)^n S_\lambda^n$,
- (ii) $\|S_\lambda e_u\|^n = \|S_\lambda^n e_u\|$ for all $u \in V$.

The basic facts on bounded composition operators in L^2 -spaces we need in this paper can be found in [14] (see also [3] for the case of unbounded composition operators).

3. Transcendentality of $\ln(\alpha)$

The irrationality of e was established by Euler in 1744 and that of π was proven by Johann Heinrich Lambert in 1761. Their transcendence was proved about a century later by Hermite and Lindemann respectively. A generalisation of the above result was given by Weierstrass in 1885, and is as follows.

THEOREM 3.1. [1, Theorem 1.4] (*Lindemann-Weierstrass theorem*). *For any finite system of distinct algebraic numbers $\alpha_1, \dots, \alpha_n$, the numbers $e^{\alpha_1}, \dots, e^{\alpha_n}$ are linearly independent over \mathbb{A} .*

For the reader's convenience, we include the proof of the following result which is surely folklore. This fact will be used in Section 5.

COROLLARY 3.2. *$\ln(\alpha)$ is transcendental for any algebraic number $\alpha \neq 0, 1$.*

PROOF. Suppose that, contrary to our claim, $\ln(\alpha)$ is algebraic. Then, by Theorem 3.1 with $\alpha_1 = 0$ and $\alpha_2 = \ln(\alpha)$, we see that 1 and $\alpha = e^{\ln(\alpha)}$ are linearly independent over \mathbb{A} , which gives a contradiction. \square

4. Bounded classical weighted shifts

In this section, we will show that for every integer n greater than or equal to 2, any bounded bilateral weighted shift W satisfying the equation $(W^*W)^n = (W^*)^n W^n$ is quasinormal (to simplify terminology, we drop the adjective “classical” in this section). We begin by proving two key lemmata.

LEMMA 4.1. *Let $k \in \mathbb{N}$. Then each root of the polynomial*

$$(4.1) \quad p(z) = kz^k - (z^{k-1} + z^{k-2} + \dots + 1), \quad z \in \mathbb{C},$$

except for $z = 1$, is in the open unit disk centered at 0. Moreover, all roots of the polynomial $p(z)$ are of multiplicity one.

PROOF. It is enough to consider the case of $k \geq 2$. Suppose z is a complex number such $|z| > 1$ and $p(z) = 0$. Then $|z^k| > |z^i|$ for $i = 0, 1, \dots, k-1$. This implies that

$$|kz^k| > |z^{k-1} + z^{k-2} + \dots + 1| = |kz^k|,$$

which gives a contradiction. This means that all roots of the polynomial $p(z)$ satisfies the inequality $|z| \leq 1$.

It is clear that $p(1) = 0$. We show that 1 is in fact the only root which lies on the unit circle. Suppose that there exists $z \in \mathbb{C}$ such that $|z| = 1$, $p(z) = 0$ and $z \neq 1$. Since the polynomial $p(z)$ has real coefficients, $\bar{z} = \frac{1}{z}$ is its root as well. Hence we have

$$(4.2) \quad 0 = kz^k - (z^{k-1} + \dots + 1),$$

$$(4.3) \quad 0 = k\left(\frac{1}{z}\right)^k - \left(\left(\frac{1}{z}\right)^{k-1} + \dots + 1\right).$$

It follows from (4.3) that

$$k\frac{1}{z} = 1 + z + \dots + z^{k-1}.$$

This and (4.2) yield

$$(4.4) \quad z^{k+1} = 1.$$

On the other hand multiplying both sides of the equality in (4.2) by $z - 1$ we get

$$(4.5) \quad kz^{k+1} - (k+1)z^k + 1 = 0.$$

Applying (4.4) and (4.5), we see that $z^k = 1$ and so, by (4.4), $z = 1$. This contradicts the assumption that $z \neq 1$.

Now we will prove the “moreover” part of the theorem. Using [7, Theorem III.6.10], we easily verify that 1 is not a multiple root of the polynomial $p(z)$. Suppose that the polynomial $p(z)$ has a multiple root different from 1. Then clearly the polynomial $q(z) = (z - 1)p(z)$ has a multiple root different from 1. Applying [7, Theorem III.6.10] again, we deduce that the polynomials $q(z)$ and $q'(z)$ has a

common root different from 1. Since the polynomials $q(z) = kz^{k+1} - (k+1)z^k + 1$ and $q'(z) = k(k+1)z^{k-1}(z-1)$ have only one common root 1, we get a contradiction. \square

LEMMA 4.2. *Let $k \in \mathbb{N}$. Suppose that $\{a_n\}_{n=-\infty}^{\infty}$ is a bounded sequence of real numbers that satisfies the following recurrence relation*

$$ka_n = a_{n+1} + \dots + a_{n+k}, \quad n \in \mathbb{Z}.$$

Then $\{a_n\}_{n=-\infty}^{\infty}$ is a constant sequence.

PROOF. Without loss of generality, we can assume that $k \geq 2$. Suppose that, contrary to our claim, the sequence $\{a_n\}_{n=-\infty}^{\infty}$ is not constant. Then there exist $r \in \mathbb{Z}$ and $\varepsilon > 0$ such that

$$(4.6) \quad |a_r - a_{r-1}| > \varepsilon.$$

Given $l \in \mathbb{N}$, we define the sequence $\{b_n^{(l)}\}_{n=0}^{\infty}$ by

$$(4.7) \quad b_n^{(l)} = a_{l-n}, \quad n \in \mathbb{Z}_+.$$

Clearly, $\{b_n^{(l)}\}_{n=0}^{\infty}$ satisfies the recurrence relation

$$(4.8) \quad kb_{n+k}^{(l)} = b_n^{(l)} + \dots + b_{n+k-1}^{(l)}, \quad n \in \mathbb{Z}_+,$$

with the initial values $b_0^{(l)} = a_l, \dots, b_{k-1}^{(l)} = a_{l-k+1}$. The polynomial $\frac{1}{k}p(z)$, where $p(z)$ is as in (4.1), is the characteristic polynomial of the recurrence relation (4.8). By Lemma 4.1 and [6, Theorem 3.1.1], we have

$$(4.9) \quad b_n^{(l)} = A_1^{(l)} z_1^n + \dots + A_k^{(l)} z_k^n, \quad n \in \mathbb{Z}_+,$$

where z_1, \dots, z_k are the roots of the polynomial $\frac{1}{k}p(z)$, and $A_1^{(l)}, \dots, A_k^{(l)}$ are complex numbers depending on the initial values $b_0^{(l)}, \dots, b_{k-1}^{(l)}$. In view of Lemma 4.1, we may assume that $z_1 = 1$. It follows from (4.9) that $A_1^{(l)}, \dots, A_k^{(l)}$ is a solution of the system of linear equations in unknowns w_1, \dots, w_k :

$$(4.10) \quad \begin{cases} b_0^{(l)} = w_1 + \dots + w_k \\ b_1^{(l)} = w_1 z_1 + \dots + w_k z_k \\ \vdots \\ b_{k-1}^{(l)} = w_1 z_1^{k-1} + \dots + w_k z_k^{k-1}. \end{cases}$$

Let U be the matrix associated with the system (4.10), i.e.,

$$U = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_k \\ \vdots & \vdots & \dots & \vdots \\ z_1^{k-1} & z_2^{k-1} & \dots & z_k^{k-1} \end{bmatrix}.$$

Since U is a Vandermonde matrix, we deduce that $\det U \neq 0$. Hence the system (4.10) has a unique solution which, by Cramer's Rule (cf. [7, Corollary VII.3.8]), is given by

$$(4.11) \quad A_j^{(l)} = \frac{\det U_j^{(l)}}{\det U}, \quad j = 1, \dots, k,$$

where $U_j^{(l)}$ is the matrix formed by replacing the j th column of U by the transpose of the row vector $[b_0^{(l)}, b_1^{(l)}, \dots, b_{k-1}^{(l)}] = [a_l, a_{l-1}, \dots, a_{l-k+1}]$.

By assumption, $C := \sup\{|a_n| : n \in \mathbb{Z}\} < \infty$. Set $L = k!C$. Note that

$$(4.12) \quad |\det U_j^{(l)}| \leq L, \quad j = 1, 2, \dots, k.$$

Indeed, this can be deduced by estimating each summand of $\det U_j^{(l)}$ (cf. [7, Theorem VII.3.5]) and using the fact that $z_1 = 1$ and $p := \max\{|z_i| : i = 2, \dots, k\} < 1$ (cf. Lemma 4.1). Take $l \in \mathbb{N}$ such that $l \geq r$ and $2(k-1)p^{l-r} \frac{L}{|\det U|} < \varepsilon$. This combined with (4.11) and (4.12) yields

$$\begin{aligned} |a_r - a_{r-1}| &\stackrel{(4.7)}{=} |b_{l-r}^{(l)} - b_{l-r+1}^{(l)}| \\ &\stackrel{(4.9)}{=} |A_1^{(l)} z_1^{l-r} + \dots + A_k^{(l)} z_k^{l-r} - (A_1^{(l)} z_1^{l-r+1} + \dots + A_k^{(l)} z_k^{l-r+1})| \\ &= |A_2^{(l)} z_2^{l-r} + \dots + A_k^{(l)} z_k^{l-r} - (A_2^{(l)} z_2^{l-r+1} + \dots + A_k^{(l)} z_k^{l-r+1})| \\ &\leq 2(k-1)p^{l-r} \frac{L}{|\det U|} < \varepsilon, \end{aligned}$$

which contradicts (4.6). This completes the proof. \square

Now we are ready to prove the main result of this section. Recall the well-known and easy to prove fact that a quasinormal injective bilateral weighted shift is a multiple of a unitary operator.

THEOREM 4.3. *Let $k \geq 2$. Then any bounded injective bilateral weighted shift W that satisfies the equality $(W^*W)^k = (W^*)^k W^k$ is quasinormal.*

PROOF. Let W be a bounded injective bilateral weighted shift with weights $\{\lambda_n\}_{n=-\infty}^\infty$ (cf. (2.1)). Without loss of generality, we can assume that $\lambda_n > 0$ for all $n \in \mathbb{Z}$ (cf. [15]). Suppose that $(W^*W)^k = (W^*)^k W^k$ for some $k \geq 2$. By Proposition 2.3, we have

$$(4.13) \quad \lambda_n^k = \lambda_n \lambda_{n+1} \cdots \lambda_{n+k-1}, \quad n \in \mathbb{Z}.$$

Since that operator W is bounded, the sequence $\{\lambda_n\}_{n=-\infty}^\infty$ is bounded as well. We will show that there exists $c \in (0, \infty)$ such that $\lambda_n > c$ for every $n \in \mathbb{N}$. If not, there exists a subsequence $\{\lambda_{n_i}\}_{i=1}^\infty$ of $\{\lambda_n\}_{n=1}^\infty$ such that $\lambda_{n_i} \rightarrow 0$ as $i \rightarrow \infty$. Set

$$(4.14) \quad d = \min_{i=1, \dots, k} \lambda_i$$

and

$$D = \sup_{i \in \mathbb{Z}} \lambda_i.$$

Then there exists $m \in \mathbb{N}$ such that $n_m > k$ and $\lambda_{n_m} < \frac{d^k}{D^{k-1}}$. By (4.13) we have

$$\lambda_{n_m-i}^k = \lambda_{n_m-i} \lambda_{n_m-i+1} \cdots \lambda_{n_m-i+k-1} < \frac{d^k}{D^{k-1}} D^{k-1} = d^k, \quad i = 0, 1, \dots, k-1.$$

Hence

$$(4.15) \quad \lambda_{n_m-i} < d, \quad i = 0, 1, \dots, k-1.$$

Since each term of the sequence $\{\lambda_i\}_{i=-\infty}^{n_m-k}$ is a geometric mean of $k-1$ positive real numbers smaller than d , we deduce from (4.13) and (4.15) that $\lambda_n < d$ for every $n \leq n_m$. In particular, $\lambda_i < d$ for $i = 1, 2, \dots, k$, which contradicts (4.14). Applying (4.13) again, we easily see that the sequence $\{\lambda_i\}_{i=-\infty}^0$ is bounded below from zero. Altogether this implies that the whole sequence $\{\lambda_i\}_{i=-\infty}^\infty$ is bounded below from zero.

Now we define the sequence $\{a_n\}_{n=-\infty}^{\infty}$ by $a_n = \log \lambda_n$. In view of the previous paragraph, the sequence $\{a_n\}_{n=-\infty}^{\infty}$ is bounded. It follows from (4.13) that $\{a_n\}_{n=-\infty}^{\infty}$ satisfies the following recurrence relation

$$(k-1)a_n = a_{n+1} + \dots + a_{n+k-1}, \quad n \in \mathbb{Z}.$$

Hence, by Lemma 4.2, the sequence $\{a_n\}_{n=-\infty}^{\infty}$ is constant. It is easily seen that W is a multiple of a unitary operator and as such is quasinormal. This completes the proof. \square

It is worth pointing out that Theorem 4.3 is no longer true if the bilateral weighted shift is not bounded.

EXAMPLE 4.4. Let W be an injective bilateral weighted shift with weights $\{\lambda_n\}_{n=-\infty}^{\infty}$ given by $\lambda_n = \exp((-2)^n)$ for $n \in \mathbb{Z}$. Then the sequence $\{\lambda_n\}_{n=-\infty}^{\infty}$ satisfies (4.13) with $k = 3$, but it does not satisfy (4.13) with $k = 2$. Denote by \mathcal{E} the linear span of the standard orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$ of $\ell^2(\mathbb{Z})$. Clearly $\mathcal{E} \subset \mathcal{D}(W) \cap \mathcal{D}(W^*)$, $W(\mathcal{E}) \subset \mathcal{E}$ and $W^*(\mathcal{E}) \subset \mathcal{E}$. This, by the von Neumann theorem (cf. [17, Theorem 5.3]), implies that W^3 is closable. Hence, we have

$$(W^*W)^3|_{\mathcal{E}} = W^{*3}W^3|_{\mathcal{E}} \subset W^{*3}W^3 \subset (W^3)^*W^3 \subset (\overline{W^3})^*\overline{W^3}.$$

Since \mathcal{E} is a core for the selfadjoint operator $(W^*W)^3$ (cf. [8, Proposition 3.4.3]) and $(\overline{W^3})^*\overline{W^3}$ is selfadjoint (cf. [17, Theorem 5.39]), we deduce from the maximality of selfadjoint operators that

$$(W^*W)^3 = W^{*3}W^3 = (\overline{W^3})^*\overline{W^3}.$$

It is easily seen that $(W^*W)^2 \neq W^{*2}W^2$. Hence, W is not quasinormal (cf. [10, Lemma 3.5]).

5. Weighted shifts on directed trees and composition operators

Our aim in this section is to construct for every integer $n \geq 2$ an injective non-quasinormal weighted shift $S_\lambda \in \mathcal{B}(\ell^2(V_\infty))$ on a directed tree $\mathcal{T}_\infty = (V_\infty, E_\infty)$ satisfying the condition (1.1) with $A = S_\lambda$, where $\mathcal{T}_\infty = (V_\infty, E_\infty)$ is the rootless directed tree with one branching vertex defined by

$$(5.1) \quad V_\infty = \{-k : k \in \mathbb{Z}_+\} \sqcup \{(i, j) : i, j \in \mathbb{N}\},$$

$$(5.2) \quad E_\infty = \{(-k, -k+1) : k \in \mathbb{N}\} \sqcup \{(0, (i, 1)) : i \in \mathbb{N}\} \\ \sqcup \{((i, j), (i, j+1)) : i, j \in \mathbb{N}\}.$$

(The symbol " \sqcup " denotes disjoint union of sets.) The weights of S_λ are defined with the help of three sequences $\{\alpha_i\}_{i=1}^{\infty}$, $\{\beta_i\}_{i=1}^{\infty}$ and $\{\gamma_i\}_{i=0}^{\infty}$ of positive real numbers as follows

$$(5.3) \quad \lambda_v = \begin{cases} \alpha_i & \text{if } v = (i, 1), i \in \mathbb{N}, \\ \beta_i & \text{if } v = (i, j), i \in \mathbb{N}, j \geq 2, \\ \gamma_i & \text{if } v = -i, i \in \mathbb{Z}_+. \end{cases}$$

It is a matter of routine to verify that Proposition 2.3 takes now the following form.

PROPOSITION 5.1. *If $n \geq 2$ and S_λ is a weighted shift on the directed tree \mathcal{T}_∞ with weights (5.3), then $(S_\lambda^* S_\lambda)^n = S_\lambda^{*n} S_\lambda^n$ if and only if the following three conditions hold:*

$$(i) \quad \gamma_{k+n-1}^n = \gamma_{k+n-1} \gamma_{k+n-2} \cdots \gamma_k \text{ for all } k \in \mathbb{Z}_+,$$

- (ii) $\gamma_{n-i-1}^n = \gamma_{n-i-1} \cdots \gamma_0 \sqrt{\sum_{k=1}^{\infty} \alpha_k^2 \beta_k^{2(i-1)}}$ for all $i \in \{1, 2, \dots, n-1\}$,
- (iii) $(\sqrt{\sum_{k=1}^{\infty} \alpha_k^2})^n = \sqrt{\sum_{k=1}^{\infty} \alpha_k^2 \beta_k^{2(n-1)}}$.

Our next goal is to consider a sequence $\{S_k\}_{k \in \mathbb{Z}}$ of functions on $(0, 1)$ that will play an essential role in the proof of Theorem 5.3. Given $k \in \mathbb{Z}$, we define a function $S_k : (0, 1) \rightarrow (0, \infty)$ by

$$S_k(x) = 1^k + 2^k x + 3^k x^2 + \dots, \quad x \in (0, 1).$$

The following formulas are well-known in classical analysis:

$$(5.4) \quad S_0(x) = \frac{1}{1-x} \text{ and } S_{-1}(x) = -\frac{\ln(1-x)}{x} \text{ for } x \in (0, 1).$$

Below we collect some properties of the functions $\{S_k : k \in \mathbb{Z}_+\}$.

LEMMA 5.2. *There exists a (unique) sequence $\{m_k\}_{k=0}^{\infty} \subset \mathbb{Z}[x]$ such that*

- (i) $m_0 = m_1 = 1$,
- (ii) *the degree of m_k is equal to $k-1$ for every $k \in \mathbb{N}$,*
- (iii) *the leading coefficient of m_k is equal to 1 for every $k \in \mathbb{Z}_+$,*
- (iv) *1 is not a root of m_k for every $k \in \mathbb{Z}_+$,*
- (v) $S_k(x) = \frac{m_k(x)}{(1-x)^{k+1}}$ for every $k \in \mathbb{Z}_+$.

In particular, $S_k(x) \in \mathbb{Q}(x)$ for every $k \in \mathbb{Z}_+$.

PROOF. We use induction on k . The case of $k = 0$ follows from (5.4). Suppose that $m_0, \dots, m_k \in \mathbb{Z}[x]$ have the required properties for a fixed (unspecified) $k \in \mathbb{Z}_+$. Note that

$$(5.5) \quad S_{k+1}(x) = (x(1^k + 2^k x + 3^k x^2 + \dots))' = (xS_k(x))', \quad x \in (0, 1).$$

By the induction hypothesis and (5.5), we have

$$(5.6) \quad S_{k+1}(x) = (xS_k(x))' = \left(\frac{xm_k(x)}{(1-x)^{k+1}} \right)' = \frac{m_{k+1}(x)}{(1-x)^{k+2}}, \quad x \in (0, 1),$$

with

$$(5.7) \quad m_{k+1}(x) := (xm_k'(x) + m_k(x))(1-x) + (k+1)xm_k(x), \quad x \in (0, 1).$$

Now, it is easily seen that $m_{k+1} \in \mathbb{Z}[x]$, the leading coefficients of m_{k+1} and m_k coincide and the degree of m_{k+1} is equal to k . Suppose that, contrary to our claim, $x_0 = 1$ is a root of m_{k+1} . Then, by (5.7), we have

$$0 = m_{k+1}(1) = (k+1)m_k(1),$$

which contradicts $m_k(1) \neq 0$. Hence the fraction appearing on the right-hand side of (5.6) is irreducible. This also proves the uniqueness of m_{k+1} . \square

We are now ready to construct a weighted shift on a directed tree and a composition operator C in an L^2 -space with the properties mentioned in Introduction.

THEOREM 5.3. *Let n be an integer greater than or equal to 2. Then there exists an injective non-quasinormal weighted shift $S_\lambda \in \mathbf{B}(\ell^2(V_\infty))$ on the directed tree \mathcal{T}_∞ which satisfies the condition (1.1) with $A = S_\lambda$. Moreover, there exists an injective non-quasinormal composition operator C in L^2 -space over a σ -finite measure space satisfying the condition (1.1) with $A = C$.*

PROOF. By [9, Lemma 4.3.1], every weighted shift on a rootless and leafless directed tree with positive real weights is unitarily equivalent to an injective composition operator in an L^2 -space over a σ -finite measure space. Hence, it is enough to construct a weighted shift on a rootless and leafless directed tree with positive real weights which satisfies the condition (1.1) with $A = S_\lambda$. Let \mathcal{T}_∞ be the directed tree as in (5.1) and (5.2), and let S_λ be a weighted shift on \mathcal{T}_∞ with weights as in (5.3), where the sequences $\{\alpha_k\}_{k=1}^\infty$ and $\{\beta_k\}_{k=1}^\infty$ are given by

$$(5.8) \quad \alpha_k = \sqrt{k^{n-1}q^{k-1}} \text{ and } \beta_k = \sqrt{\frac{1}{k}c^{\frac{1}{n-1}}} \text{ for } k \in \mathbb{N};$$

here the numbers q and c are chosen to satisfy the following three conditions

$$(5.9) \quad q \in \mathbb{Q} \cap (0, 1) \text{ and } c \in \mathbb{Q} \cap (0, \infty),$$

$$(5.10) \quad (S_{n-1}(q))^n = cS_0(q),$$

$$(5.11) \quad c^{\frac{k}{n-1}} \notin \mathbb{Q}, \quad k \in \{1, 2, \dots, n-2\}.$$

(Note that the condition (5.11) is empty when $n = 2$.) Below, we will show how to construct such c and q . Let $\{\gamma_k\}_{k=0}^\infty$ be a sequence of positive real numbers uniquely determined by the recurrence formulas (i) and (ii) of Proposition 5.1. Since, by (5.8), the following equalities hold

$$(5.12) \quad \sum_{k=1}^{\infty} \alpha_k^2 = S_{n-1}(q),$$

$$(5.13) \quad \sum_{k=1}^{\infty} \alpha_k^2 \beta_k^{2(i-1)} = c^{\frac{i-1}{n-1}} S_{n-i}(q), \quad i \in \mathbb{Z},$$

one can infer from (5.9) and Lemma 5.2 that $\{\gamma_k\}_{k=0}^\infty$ is sequence of algebraic numbers. According to Proposition 5.1(i), for every integer $i \geq n-1$, the term γ_i is a geometric mean of $n-1$ preceding terms $\gamma_{i-1}, \dots, \gamma_{i-n+1}$. Hence, the sequence $\{\gamma_i\}_{i=0}^\infty$ is bounded (see the proof of Theorem 4.3). This combined with (5.12), (5.8) and Proposition 2.2 shows that $S_\lambda \in \mathbf{B}(\ell^2(V_\infty))$.

Now we prove that there exist c and q satisfying (5.9), (5.10) and (5.11). For this we consider a new quantity $c_0 \in (0, \infty)$ uniquely determined by the equation $c = c_0(S_{n-1}(q))^{n-1}$. Then the equality (5.10) takes the form $S_{n-1}(q) = c_0 S_0(q)$, and thus by (5.4) and Lemma 5.2 we have

$$(5.14) \quad c_0 = \frac{S_{n-1}(q)}{S_0(q)} = \frac{m_{n-1}(q)}{(1-q)^{n-1}}.$$

It is easily seen that c satisfies (5.9), (5.10) and (5.11) if and only if c_0 does. Hence, it remains to construct q and c_0 . We may assume that $n \geq 3$. Let p be a prime number and let $m_{n-1}(x) = a_{n-2}x^{n-2} + \dots + a_0$ with $a_0, \dots, a_{n-2} \in \mathbb{Z}$. Set $q = \frac{1}{p}$. Then, by (5.14), we have

$$c_0 = \frac{p(a_0p^{n-2} + \dots + a_{n-2})}{(p-1)^{n-1}}.$$

Since, by Lemma 5.2(iii), $a_{n-2} = 1$ and p is prime, an elementary reasoning shows that the number c_0 satisfies (5.11). This gives the required q and c .

To complete the proof, it suffices to consider tree disjunctive cases.

CASE 1. $(S_\lambda^* S_\lambda)^p \neq S_\lambda^{*p} S_\lambda^p$ for every $p \in \{2, 3, \dots, n-1\}$.

Indeed, otherwise $(S_\lambda^* S_\lambda)^p = S_\lambda^{*p} S_\lambda^p$ for some $p \in \{2, 3, \dots, n-1\}$. In view of Proposition 5.1(iii), we have

$$\left(\sum_{k=1}^{\infty} \alpha_k^2\right)^p = \sum_{k=1}^{\infty} \alpha_k^2 \beta_k^{2(p-1)},$$

which together with (5.12) and (5.13) yields

$$(5.15) \quad S_{n-1}(q)^p = c^{\frac{p-1}{n-1}} S_{n-p}(q).$$

It follows from (5.9) and Lemma 5.2 that $S_{n-1}(q) \in \mathbb{Q}$ and $S_{n-p}(q) \in \mathbb{Q}$. This and (5.15) lead to a contradiction with (5.11).

CASE 2. $(S_\lambda^* S_\lambda)^p \neq S_\lambda^{*p} S_\lambda^p$ for $p = n+1$.

Indeed, otherwise arguing as in Case 1 we show that

$$(5.16) \quad S_{n-1}(q)^{n+1} = c^{\frac{n}{n-1}} S_{-1}(q).$$

Since the numbers $S_{n-1}(q)$ and $c^{\frac{n}{n-1}}$ are algebraic, the equality (5.16) leads to a contradiction with the fact that the number $S_{-1}(q) = -\frac{\ln(1-q)}{q}$ is transcendental (see (5.4) and Corollary 3.2).

CASE 3. $(S_\lambda^* S_\lambda)^p \neq S_\lambda^{*p} S_\lambda^p$ for every $p \in \{n+2, n+3, \dots\}$.

Indeed, otherwise $(S_\lambda^* S_\lambda)^p = S_\lambda^{*p} S_\lambda^p$ for some $p \in \{n+2, n+3, \dots\}$. Hence, by Proposition 5.1(ii) and (5.13), we have

$$(5.17) \quad \gamma_{p-i-1}^{2p} = \gamma_{p-i-1}^2 \cdots \gamma_0^2 c^{\frac{i-1}{n-1}} S_{n-i}(q), \quad i \in \{1, 2, \dots, p-1\}.$$

Substituting $i = n+1$ into (5.17), we get

$$\gamma_{p-n-2}^{2p} = \gamma_{p-n-2}^2 \cdots \gamma_0^2 c^{\frac{n}{n-1}} S_{-1}(q).$$

Since $\{\gamma_k\}_{k=0}^\infty$ is sequence of algebraic numbers, we can argue as in Case 2 to get a contradiction with the transcendental of $S_{-1}(q)$. This completes the proof. \square

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